Short Papers

Unloaded Q Measurement—The Critical-Points Method

En-Yuan Sun and Shuh-Han Chao

Abstract— **The unloaded quality factor of resonators whether coupled magnetically or electrically, with loss or without loss, is estimated by the *critical points* (i.e., *extreme-reactance/susceptance points*) method. The *critical-points* method derived from this paper is a fast and accurate method for unloaded Q measurement and is suitable for general external coupling environment.**

I. Introduction

The unloaded quality factor Q_0 is an important figure since it establishes an upper limit for the overall device performance. A number of Q-factor measurement methods are published, and some good surveys are in [1], [2]. Unfortunately, a simple method for the measurement of Q_0 is not widely available. The method introduced by Kajfez and Hwan [3] is convenient in some cases but may not be suitable for *lossy* external-coupled circuit or for a very small impedance-locus circle of *weak undercoupled* cases when the phase data are not accurate. The measurement method to be derived and described is convenient and useful for universal cases including *external lossy* and *weak undercoupled* cases.

II. Circuit Model and Impedance Locus

A. Circuit Modeling in the Vicinity of Resonant Frequency

It is well known that the equivalent circuits showing frequency characteristics of one port can be represented by the Foster types [4], and that the equivalent circuit for input impedance or admittance of the resonant cavity may be shown in Fig. 1(a) for first Foster form or Fig. 1(b) for second Foster form, respectively. These equivalent circuits may be representative of the *detuned* position near either the *short* point (magnetical coupling predominantly) or the *open* point (electrical coupling predominantly), respectively.

B. Impedance Locus in the Vicinity of Resonant Frequency

Take the first-Foster type circuit, for example. The input impedance can be represented as follows:

$$Z_l(\omega) = R_e + j\omega L_e + \frac{R_0'}{\left(1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)} \quad (1)$$

where $R_0' = n^2 R_0$, $Q_0 = R_0\sqrt{C_0/L_0}$, $\omega_0 = 1/\sqrt{C_0 L_0}$ and R_0, L_0, C_0 are the *interior* parameters; R_e and L_e are the *exterior* parameters; and n is the *turns ratio* with interior circuit transformed to exterior circuit through coupling structures. All these parameters are normalized to the transmission-line characteristic impedance R_c for the simplification's sake. For simplicity, we use the following representation in the vicinity of the natural resonant angular frequency ω_0:

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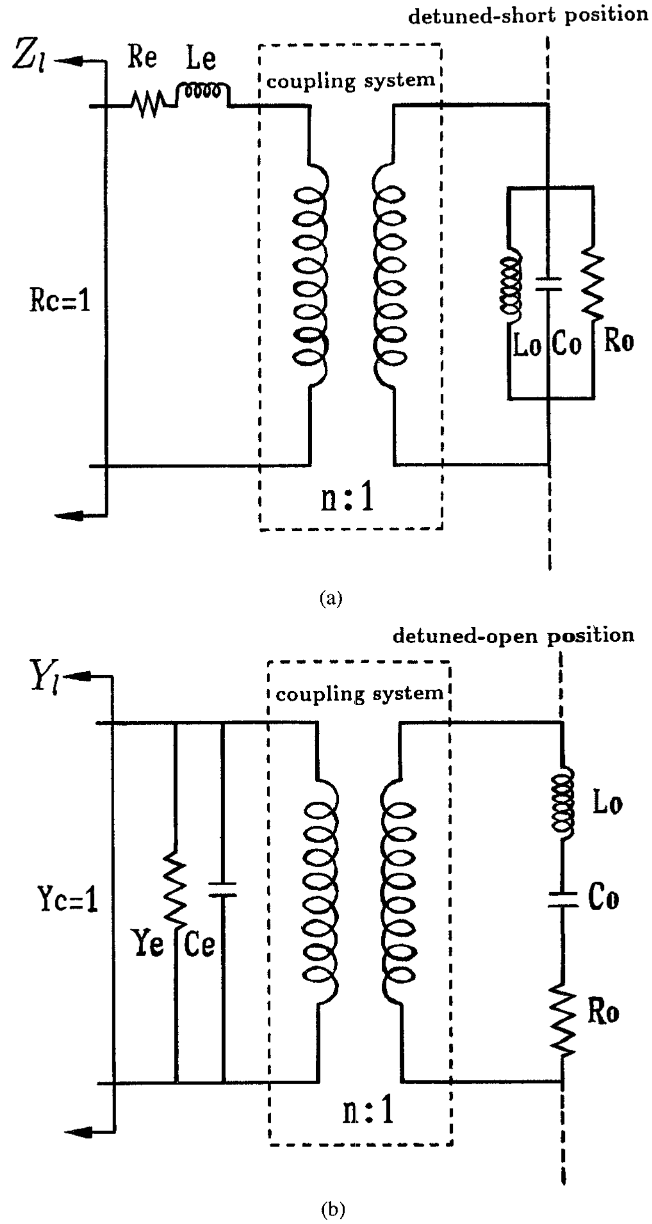


Fig. 1. Equivalent circuits of (a) first Foster form and (b) second Foster form including exterior coupling losses.

$$\frac{\omega_k}{\omega_0} - \frac{\omega_0}{\omega_k} = 2\delta_k \frac{(1 + \frac{\delta_k}{2})}{(1 + \delta_k)} = 2\delta_k D_k \quad (2)$$

$$\approx 2\delta_k \qquad \text{for } \delta_k \ll 1 \quad (3)$$

where, $\delta_k = (\omega_k - \omega_0)/\omega_0$, *the frequency-tuning parameter*; and $D_k = (1 + (\delta_k/2))/(1 + \delta_k)$, *the deviation factor of linearity defined here*. Thus, the input impedance can be represented by

$$Z_l(\omega_k) = \left[R_e + \frac{R_0'}{1 + (2Q_0\delta_k D_k)^2}\right] + j\left[\omega_0(1 + \delta_k)L_e - \frac{2Q_0\delta_k D_k R_0'}{1 + (2Q_0\delta_k D_k)^2}\right] \quad (4)$$

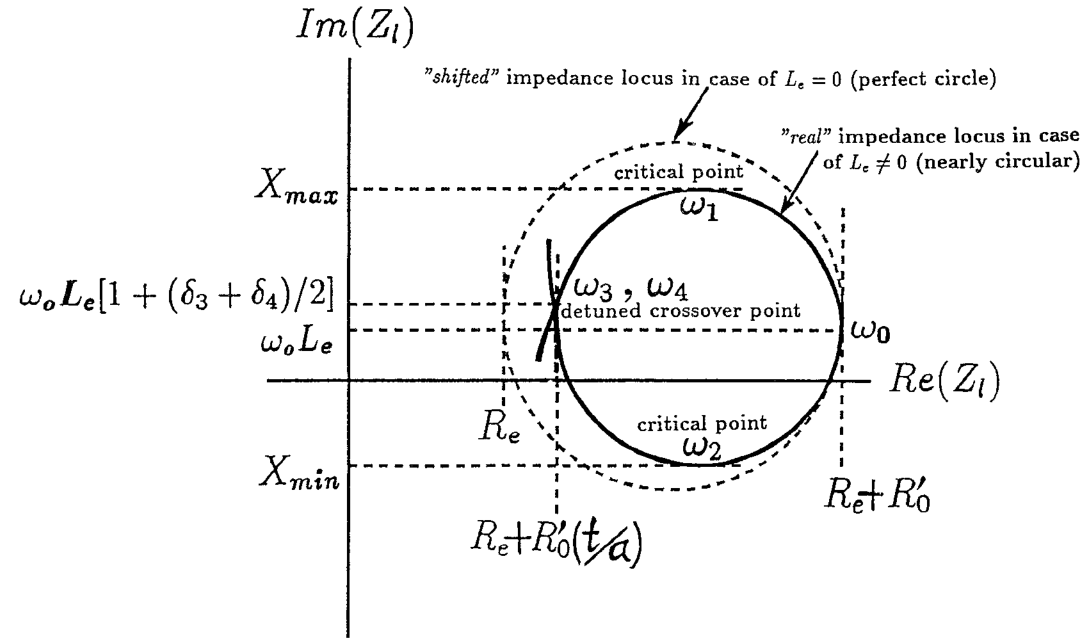


Fig. 2. The *real* input-impedance locus of magnetic coupled resonator including coupling losses and reactance in the vicinity of resonant frequency.

$$Z_l(\omega_k) = R_e + \frac{R_0'}{1 + (2Q_0\delta_k)^2} + j\left[\omega_0(1+\delta_k)L_e - \frac{2Q_0\delta_k R_0'}{1 + (2Q_0\delta_k)^2}\right] \quad \text{for } \delta_k \ll 1. \tag{5}$$

At the resonant frequency, $Z_l(\omega_0) = (R_e + R_0') + j\omega_0 L_e$ from (1), and at the *detuned crossover point* there are two angular frequencies ω_3 and ω_4 on Fig. 2 where $Z_l(\omega_3) = Z_l(\omega_4)$, i.e.,

$$\frac{R_0'}{1 + (2Q_0\delta_3 D_3)^2} = \frac{R_0'}{1 + (2Q_0\delta_4 D_4)^2} \tag{6}$$

$$\omega_0\delta_3 L_e - \frac{2Q_0\delta_3 D_3 R_0'}{1 + (2Q_0\delta_3 D_3)^2} = \omega_0\delta_4 L_e - \frac{2Q_0\delta_4 D_4 R_0'}{1 + (2Q_0\delta_4 D_4)^2}. \tag{7}$$

In solving (6) and (7) simultaneously without regard to the trivial solution of $\omega_3 = \omega_4$, we get

$$\delta_3 D_3 = -\delta_4 D_4 \tag{8}$$

$$t/a = \frac{1}{1 + (2Q_0\delta_3 D_3)^2} \tag{9}$$

where, $t = \omega_0 L_e/(2Q_0 R_0')$; and $a = 2D_3/(1 + (D_3/D_4))$. Also, $Z_l(\infty) = R_e + j\infty$ and $Z_l(0) = R_e$ is established only when the single-resonant model is satisfied. However, since ω_k is far away from the *detuned crossover point*, the dominant factor should be switched to another resonant model and the values $Z_l(\infty)$ and $Z_l(0)$ evaluated from (1) become inaccurate. Then, the impedance locus around the central frequency ω_0 may have a resemblance to that shown in Fig. 2. For the sake of comparison, we also show the *shifted* ideal impedance-locus circle in the case without exterior inductance, the circle of which is shifted from the $\text{Im}(Z_l) = 0$ axis to the $\text{Im}(Z_l) = \omega_0 L_e$ axis.

From Fig. 2, we can see the *real and nearly circular* locus is deformed from the *ideal and circular* one. The factor t/a can be called the *factor of circle deformation*. It is easy to show that the smaller the factor, the closer the real impedance locus is to a circle.

III. Derivation and Measurement Procedure of the Critical-Points Method

From nearly circular impedance locus, we can get two points corresponding to two extreme values, maximum and minimum, of reactance within this circle. The corresponding frequencies ω_1 and ω_2 are both shown in Fig. 2 and should be in the vicinity of the resonant frequency for high Q_0; thus (3) and (5) can be used to find these *extreme frequency-tuning parameters*, δ_1, and δ_2. Taking the first-order partial derivatives with respect to δ_k in the second term of (5) (i.e., reactance of the input impedance) and then finding points where the partial derivative is equal to zero, we get

$$\frac{\partial \text{Im}[Z_l(\omega_0(1+\delta_k))]}{\partial \delta_k} = 2Q_0 R_0' \left\{ t + \frac{2(2Q_0\delta_k)^2}{[1 + (2Q_0\delta_k)^2]^2} - \frac{1}{1 + (2Q_0\delta_k)^2} \right\} = 0. \tag{10}$$

Points that satisfy (10) are known as *critical points* [6], and therefore this method derived here is named as the *critical-points method*.

Let $2Q_0\delta_k = x$, $(\delta_3 D_3/\delta_k)^2 = b$, and substitute x, b into (10) and (9), respectively. We can get

$$\frac{1 - x^2}{(1 + x^2)^2} = t = \frac{a}{1 + bx^2} \tag{11}$$

The solution is

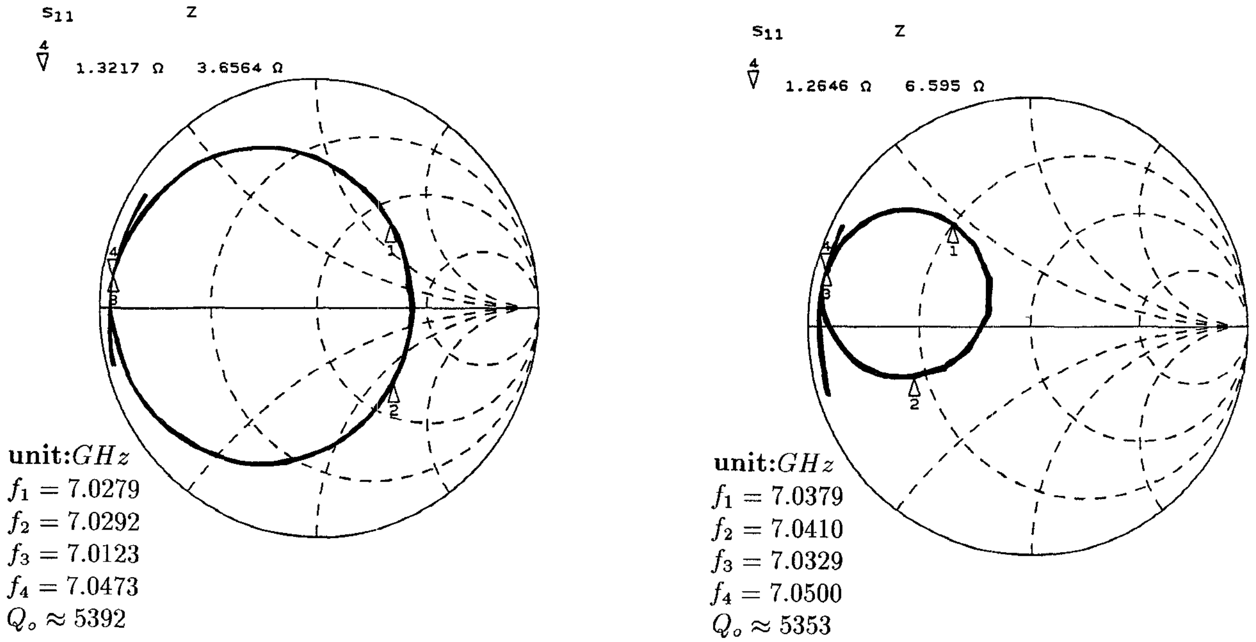
$$x^2 = \frac{(b - 2a - 1) \pm \sqrt{(b - 2a - 1)^2 - 4(b + a)(a - 1)}}{2(b + a)} \tag{12}$$

The sign "+" is the only reasonable solution to (12). After these values of a, b, and x have been solved, by the definition of

$$Q_0 = \frac{|x|}{2|\delta_k|} \tag{13}$$

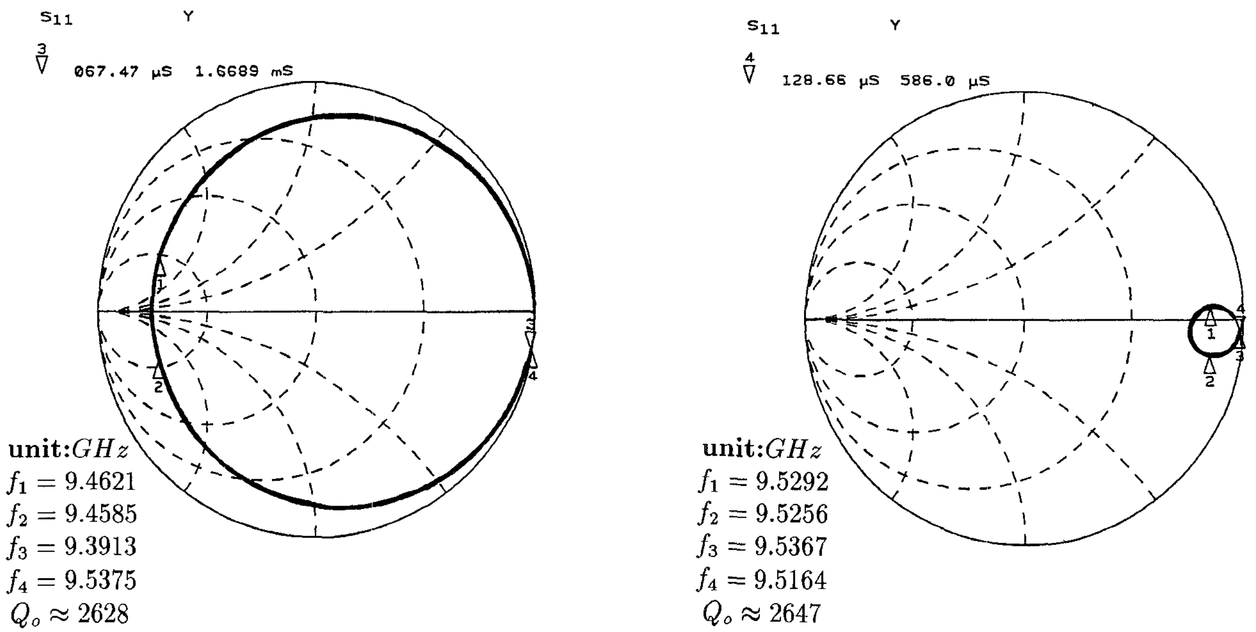
the *frequency-tuning parameters*, δ_1 and δ_2 corresponding to two critical points, are related by

$$\delta_1 = -\delta_2 \tag{14}$$



(a) (b)

Fig. 3. Measured input reflection coefficient of the dielectric resonator under (a) *overcoupled* and (b) *undercoupled* conditions (Smith Chart for impedance).



(a) (b)

Fig. 4. Measured input reflection coefficient of the metal cavity under (a) *overcoupled* and (b) *undercoupled* conditions (inverted Smith chart for impedance).

The ω_0 can also be evaluated by $\omega_0 = (\omega_1+\omega_2)/2$ and the *bandwidth of the critical points* is $2\omega_0|\delta_k| = |\omega_1 - \omega_2|$. Alternatively, the Q_0 can be written as follows:

$$Q_0 = \frac{\omega_1 + \omega_2}{2|\omega_1 - \omega_2|}|x| \tag{15}$$

$$\approx \frac{f_1 + f_2}{2|f_1 - f_2|} \quad \text{for} \quad |x| \approx 1. \tag{16}$$

To reduce random error due to measurement, by using $|\delta_1| = |\delta_2| = |\delta_2 - \delta_1|/2$ and $|\delta_3 D_3| = |\delta_4 D_4| = |\delta_4 D_4 - \delta_3 D_3|/2$, a and b can be averaged and represented by the following:

$$a = 1 + \frac{f_1^2 + f_2^2 + 2f_1 f_2 - 4f_3 f_4}{2(f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + 4f_3 f_4)} \tag{17}$$

$$b = \left(\frac{\delta_4 D_4 - \delta_3 D_3}{\delta_2 - \delta_1}\right)^2 = \left(\frac{f_4 - f_3 - \left(\frac{f_1+f_2}{2}\right)^2\left(\frac{1}{f_4} - \frac{1}{f_3}\right)}{2(f_2 - f_1)}\right)^2. \tag{18}$$

In the special case, e.g. if $R_e = L_e = 0$, then $a = 1, b = \infty, |x| = 1$

and $Q_0 = 1/(2|\delta_k|)$, which can be shown that the critical points in this case are those located at $\text{Re}(Z_l) = \text{Im}(Z_l)$, and this corresponds exactly to the argument developed in [1], [5]. Here, we may name $|x|$ as the *modification factor of critical-points bandwidth*. Usually, this factor is close to one and can be ignored in most cases [7]. For the sake of measurement, we can avail ourselves of the vector network analyzer to measure input impedance in the sweep-frequency operation mode, then mark the *critical-points frequencies* f_1 and f_2 in which the corresponding *reactance* is maximum in f_1 or f_2 and minimum *vice versa* only within the impedance locus, and finally mark the *detuned crossover frequencies* f_3 and f_4 in which the corresponding *impedance* is identical with crossover point. Thus, Q_0 can be estimated from (15). Moreover, even the simpler (16) is accurate enough in most cases. For the measured admittance locus near the *detuned-open point*, which is electrical coupling predominantly, the second-Foster type equivalent circuit such as Fig. 1(b) can be used. Follow the same derivation process as mentioned above. However, the *impedance* locus and parameters are changed to the *admittance* locus and parameters for the *dual* expression, respectively. Similar results can be developed and the *inverted* Smith Chart can be used for measurement.

IV. Experimental Results

To illustrate the principle and procedure, two types of resonators, dielectric resonator and metal cavity, were measured. In the case of the overcoupled dielectric resonator, the measured input impedance is shown in Fig. 3(a) and the frequencies for critical points and detuned crossover point are: f_1 =7.0279 GHz, f_2 =7.0292 GHz, f_3 =7.0123 GHz, and f_4 =7.0473 GHz, respectively. Thus, Q_0 =5392 is estimated from (15), Q_0 =5407 from (16), and $|x|$ =0.997 from (12). These estimated Q_0 corresponds to the result of Kajfez's method, which gets Q_0 ≈5400, and the difference can be negligible if measured error is considered. Fig. 3(b) is the measured result with the identical dielectric resonator but in the undercoupled condition, and in this case f_1 =7.0397 GHz, f_2 =7.0410 GHz, f_3 =7.0329 GHz, and f_4 =7.0500 GHz, thus Q_0 =5353 or Q_0 =5416 and $|x|$ =0.988 are estimated from (15), (16), and (12), respectively. The resultant Q_0 ≈5400 is nearly identical with the overcoupled result if measured error is considered and ignored. Fig. 4(a) and (b) are the measurement plots for the cases of the same metal cavity under overcoupled and undercoupled conditions, respectively. Again, both results get Q_0 ≈2600 by *critical-points method* with the inverted Smith Chart, and the small difference can be neglected in practice. Also note that the *modification factor of critical-points bandwidth* is $|x|$ =0.937 and the deviation error of Q_0 between (15) and (16) is less than 7% even for the case of *weak undercoupled* shown in Fig. 4(b).

V. Conclusion

The principle and measurement procedure described in this paper result in the *critical-points method*. As far as the four-frequency measurement is concerned, four frequencies of three points need measuring; that is, two *critical-points* frequencies and two *detuned crossover* frequencies on the Smith chart (impedance or inverted admittance). The unloaded quality Q_0 can be estimated from (15) quickly and accurately without any subtle discrimination between lossy and lossless, undercoupled, and overcoupled cases. As far as the two-frequency measurement is concerned, only two frequencies of two critical points need measuring. From (16), it is worth pointing out that *the unloaded Q_0 could be measured and evaluated in reflection mode as easily as its counterpart the loaded Q_L in transmission mode* (i.e., *half-power points' method*). This conclusion stands to reason in most cases as long as practical errors are ignored. Thus, only two *critical-points* frequencies and (16) need to be measured and applied without any additional information such as coupling coefficients to be measured or any auxiliary tool such as transparent Smith chart template required for the determination of Q_0, which significantly simplifies the job of Q_0 measurement.

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Further Analysis of Open-Ended Dielectric Sensors

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Abstract—**The effect on the input reflection coefficient of the dimensions of open-ended coaxial lines is investigated. Using a standard FDTD technique, the effects of variations in the flange and conductor dimensions on the reflection coefficient of a 3.6 mm coaxial line immersed in water or methanol are simulated. Simulation results are compared with measurements and previous moment method calculations. It is found that the presence or absence of a flange affects the input reflection coefficient substantially in some cases. The results also show that inversion formulas developed for lines with infinite flanges are not valid for flanges with finite radii.**

I. Introduction

An open-ended coaxial line immersed in or pressed against an unknown dielectric can serve as a dielectric sensor [1]–[3]. Inversion formulas giving the permittivity as a function of the measured reflection coefficient are required for this application. Such formulas are constructed using multiple numerical simulation results. An idealized sensor with an assumed infinite flange and inner and outer conductor inner radii corresponding to a characteristic impedance of precisely 50 Ω was previously analyzed using the moment method

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